Experimental Investigation of Structural Linear and Autoparametric Interactions Under Random Excitation

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This paper presents the results of an experimental investigation of random excitation of a nonlinear two-degree-of-freedom structural model. The normal mode frequencies of the model are adjusted to have the ratio of 2:1. This ratio meets the condition of internal resonance of the analytical model. When the first normal mode is externally excited by a band-limited random excitation, the system mean square response is found to be linearly proportional to the excitation spectral density up to a certain level above which the two normal modes exhibit discontinuity governed mainly by the internal detuning parameter and the system damping ratio. The results are completely different when the second normal mode is externally excited. For small levels of excitation spectral density the response is dominated by the second normal mode. For higher levels of excitation spectral density the first normal mode attends and interacts with the second normal mode in a form of energy exchange. A number of deviations from theoretical results are observed and discussed.

I. Introduction

THE last two decades have witnessed an increasing interest in the study of dynamic behavior of nonlinear systems under deterministic and random excitations. Under certain conditions these systems may experience complex response characteristics such as jump phenomenon, limit cycles, internal resonance, saturation phenomenon, and chaotic motion. These nonlinear phenomena have been predicted theoretically[1-3] and observed experimentally[4-6] under harmonic excitations. However, most of the predicted random response characteristics, including response stochastic stability and statistics,[7,8] have not been verified experimentally. Very few experimental investigations of random vibration of nonlinear systems have been reported in the literature.[7] The lack of experimental verification may be due to several reasons. These include difficulties in generating the same statistical properties of the random excitation as those represented theoretically, and the limitations of experimental equipment. Recently, Bolotin[9] discussed a number of experimental difficulties encountered in measurements of stochastic stability of dynamic systems.

In deterministic nonlinear vibrations, the amplitude jump, limit cycles, and parametric instability are common features of nonlinear single- and multi-degree-of-freedom systems. Parametric instability takes place when the external excitation appears as a coefficient in the homogeneous part of the equation of motion. It occurs when the excitation frequency is twice (or multiple) the system natural frequency. Internal resonance and the saturation phenomenon may occur only in nonlinear systems with more than one degree of freedom. Internal resonance implies the existence of a linear relationship between the system natural frequencies and causes nonlinear normal-mode interaction in the form of energy exchange. Under external harmonic excitation, the mode that is directly excited exhibits in the beginning the same features of a single-degree-of-freedom system response, and all other modes remain dormant. As the excitation amplitude reaches a certain critical level, the other modes become unstable and the originally excited mode reaches an upper bound. In this case the mode is said to be saturated and energy is transferred to other modes. This interesting phenomenon takes place only in systems with quadratic nonlinear coupling.

Under deterministic excitations, most nonlinear characteristics can be predicted by one of the standard techniques of nonlinear differential equations. However, aerospace structures are usually subjected to turbulent airflow, and the aeroelastician is confronted with aerodynamic loads that are random in nature. These loads vary in a highly irregular fashion and can be described in terms of statistical quantities such as means, mean squares, autocorrelation functions, and spectral density functions. Ibrahim and Roberts[10,11] and Ibrahim and Heo[12,13] considered nonlinear two-degree-of-freedom structural systems and applied Gaussian and non-Gaussian closure techniques to predict the response statistics and response stochastic stability. These studies revealed that a system with internal resonance may experience nonlinear characteristics such as autoparametric interaction. Roberts[14] conducted a series of experimental tests to measure the mean square stability boundaries of a unimodal response of a coupled two-degree-of-freedom system. Roberts reported a number of difficulties in measuring the stability boundaries. Based on the authors' experience and the work of other investigators, it is understood that experimental investigations of nonlinear random vibration is not a simple task and requires careful planning and advanced equipment preparations.

The purpose of the present paper is to conduct an experimental investigation to measure the response mean squares of a nonlinear two-degree-of-freedom structural model under band-limited random excitation. The same model was analytically examined by Haddow et al.[5] under harmonic excitation and by Ibrahim and Heo[12,13] under wideband random excitation. Agreements and disagreements with theoretical predictions will be discussed together with recommendations for future experimental work.

II. Analytical Background

The random response of a two-degree-of-freedom elastic structure has been determined analytically in Refs. 12 and 13.

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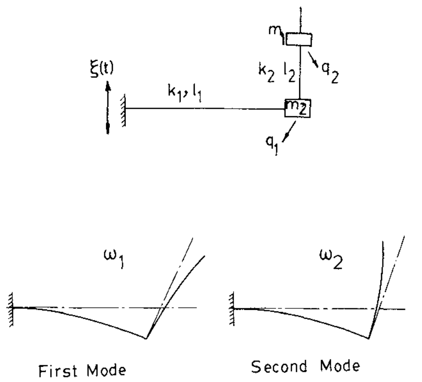


Fig. 1 Schematic diagram of the model and its first two mode shapes.

The analytical model and its first two mode shapes are shown in Fig. 1. The model consists of two beams with end masses. Under vertical support motion $\xi(t)$ the response of the two beams is mainly governed by linear dynamic and parametric couplings. However, if the system is designed such that the first two normal-mode frequencies ω_1 and ω_2 satisfy the internal resonance condition $\omega_2 = 2\omega_1$, the nonlinear inertia forces become dominant and the system dynamic response will experience different characteristics. In terms of the nondimensional normal coordinates $\boldsymbol{Y}$ the system equations of motion are

$$[I]\{Y''\} + [\zeta]\{Y'\} + [r^2]\{Y\} = \xi''(\tau)\{a\}$$
$$+ \varepsilon\xi''(\tau)[b]\{Y\} + \varepsilon\{\Psi\} \qquad (1)$$

where a prime denotes differentiation with respect to the nondimensional time parameter $\tau = \omega_1 t$, the coordinates $\boldsymbol{Y}$ are related to the dimensional normal coordinates $\boldsymbol{y}$ by the relation $\{Y_1, Y_2\} = \{y_1, y_2\}/q_1^0$, and q_1^0 is taken as the response root mean square of the system when the length of the vertical beam shrinks to zero, i.e., the response root mean square of the main beam with end mass $(m_1 + m_2)$. The elements of the vector $\{a\}$ and matrix $[b]$ are constants depending on the system properties. The small parameter ε is taken as q_1^0/ℓ_1. The matrix $[r^2]$ is diagonal with elements 1 and $(\omega_2/\omega_1)^2$. The vector $\{\Psi\}$ contains all quadratic nonlinear terms, which encompasses two groups: nonlinear terms of the same mode and autoparametric terms of the type $Y_i Y_j''$. These autoparametric terms give rise to the internal resonance condition $r = \omega_2/\omega_1 = 2$.

The random acceleration $\xi''(\tau)$ is assumed to be Gaussian wideband process with zero mean and a smooth spectral density $2D$ up to some frequency higher than any characteristic frequency of the system. The acceleration terms in the nonlinear functions Ψ_i are removed by successive elimination, and the system equations of motion is transformed into a Markov vector via the coordinates transformation

$$\{Y_1, Y_2, Y_1', Y_2'\} = \{X_1, X_2, X_3, X_4\} \qquad (2)$$

A set of first-order differential equations of the response statistical moments are generated by using the Fokker-Planck equation approach.[7] These equations are found to be coupled through higher-order moments and then closed via two approaches: Gaussian and nonGaussian closures. These closure techniques are based on the cumulant properties. The Gaussian closure is established by equating all cumulants of order greater than two to zero, i.e.,

$$\lambda_{N>2}[X_1^{k_1} X_2^{k_2} \ldots X_n^{k_n}] = 0, \qquad N = \sum_{i=1}^{n} k_i \qquad (3)$$

where λ_N is the cumulant of order N.

This approach results in 14 coupled differential equations for first- and second-order moments of the response coordinates. The numerical integration of these equations reveals that the response mean squares fluctuate between two limits. This fluctuation means that the response does not achieve a stationary state. The autoparametric interaction takes place in the neighborhood of internal resonance and is manifested by an energy exchange between the mean squares of the two normal modes.

The second method takes into account the effect of the response nonnormality. As a first-order nonGaussian approximation, all cumulants of order greater than four were equated to zero, i.e.,

$$\lambda_{N>4}[X_1^{k1} X_2^{k2} \ldots X_n^{kn}] = 0 \qquad (4)$$

This approach results in 69 first-order differential equations, in the first- through the fourth-order moments, which are solved numerically. The solution reaches a stationary state after a transient period and exhibits the same nonlinear interaction as predicted by the Gaussian closure solution. It is well known that the predicted results are approximate, and their validity has not been examined. The next section reports the measured results of a series of experimental tests of the same model under band-limited random excitation. The excitation is band-limited such that no modes higher than the first are excited.

III. Experimental Investigation

A. Experimental Model and Equipment

The experimental model is similar to a great extent to the one used by Haddow et al.[5] It consists of a horizontal beam of cross section 0.0028 × 0.0254 m and length 0.1905 m, and carries a tip mass of 0.219 Kg. The tip mass has a provision for clamping the vertical beam, which has a cross section of 0.00137 × 0.0254 m. The effective length of the vertical beam can be adjusted by changing the location of its top mass (0.185 Kg). The deflections of the two beams are measured by strain gages fixed at the root of each beam. Two gages are mounted on the horizontal beam in a two-arm bridge. Four gages are mounted on the vertical beam in a four-arm bridge. The fixed end of the horizontal beam is clamped by a fixture that is bolted on the top of the shaker armature. The shaker is a Caldyne model A88 of thrust 445N and provides ±0.0254-m peak-to-peak stroke. The shaker is powered by a Ling Electronics Model RA-250 power supply and receives a random signal through a GenRad Type 1381 Random Noise generator. The random signal is filtered to a desired bandwidth with a Krohn-Hite Model 3343 Variable Electric Filter. The filtered signal is amplified via a Calex Model 176 Instrument Amplifier. Figure 2 shows a schematic diagram of the instrumentation used in this investigation. The acceleration of the shaker platform is measured by a PCB Piezotronic Model

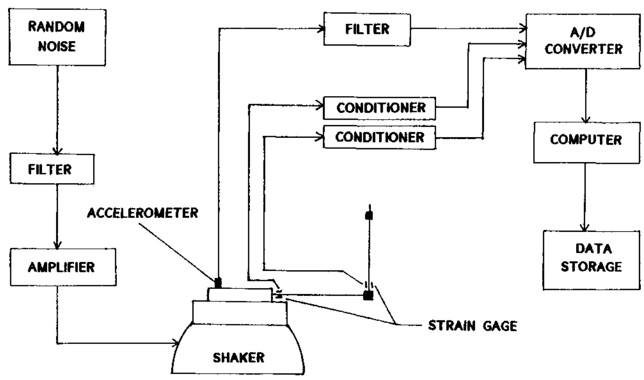


Fig. 2 Arrangement of experimental equipment.

302A02 shock accelerometer. The accelerometer is powered by a PCB Piezotronic Model 480C06 power unit.

The first two normal-mode frequencies of the system are determined theoretically and measured experimentally as a function of the beam length ratio ℓ_2/ℓ_1, as shown in Fig. 3. This figure shows that the internal resonance $\omega_2/\omega_1 = 2$ is obtained in two locations of the length ratio. At these length ratios the normal-mode frequencies are

$$\ell_2/\ell_1 = 0.485, \qquad f_1 = 9.1\ \text{Hz}, \qquad f_2 = 18.2\ \text{Hz} \quad (5a)$$

$$\ell_2/\ell_1 = 0.707, \qquad f_1 = 7.45\ \text{Hz}, \qquad f_2 = 14.9\ \text{Hz} \quad (5b)$$

The analog signals of the excitation and responses are read and converted into binary numbers using a Data Translation Model DT-3752 Intelligent Analog Peripheral (IAP). This IAP is capable of reading either eight channels (± 10 V) or 16 channels (0–10 V) of input. It can also read and convert analog signals at up to 40k points per second. This unit is mounted in an expansion slot of an IBM System 9001 Benchtop Computer. The control and programming of the Analog/Ditigal (A/D) system are accomplished through the software-controlled registers and field-selectable (hardware) options. The software controlled registers are the control registers, status/register, and gain/channel register. The control register controls the operation and mode of the A/D system. The modes used in this investigation are direct memory transfer and increment mode operation. Direct memory transfer places converted data directly into the memory of the computer. The increment mode allows the A/D to increment the input channel number automatically before each A/D conversion. This allows data to be taken from sequential channels without requiring a program to specify each channel. The status register reports the complete status of the A/D system during the operation. The gain/channel register selects the desired channels from which the data are to be taken and sets a programmable gain for all input signals. This gain is set to one for all tests. The computer controls the DT-3752 through a Fortran program. Analog signals are converted for a specified amount of time or until the computer memory is full. When the computer has completed collecting data, the data is transferred to a floppy disk for future processing.

The data processing is performed at equally spaced intervals. The problem of determining this time interval is well discussed in Bendat and Piersol.[15] Generally, if sampling is prepared at points that are too close together, it will yield correlated and redundant data. This will unnecessarily increase the labor and cost of calculation. Sampling at points that are too far apart will lead to the problem of aliasing. The aliasing is mainly a confusion between the low- and high-frequency components in the original data. In order to eliminate the problem of aliasing, a sampling rate should be chosen to be at least two times the maximum frequency that the model will experience. In order to get good sample data, a sampling rate is chosen that is roughly eight times the maximum frequency. In the present investigation, the sampling rate is chosen to be 80 Hz per channel for the first-mode excitation and 160 Hz per channel for the second-mode and wideband excitation. Data processing involves another problem known as quantization, which is the conversion of data values at the sampling points into digital form. The infinite number of values of the continuous analog signal must be approximated by a fixed set of digital levels. A choice between two consecutive levels will be required because the scale is finite. The accuracy of the approximating process is a function of the available levels, which is dependent upon the analog to digital converter resolution. The accuracy of the DT-3752 is the value of the least significant bit that corresponds to a voltage of ± 0.049 V. This resolution is analogous to a deflection of the horizontal beam of $\pm 1.85 \times 10^{-5}$ m and the vertical of $\pm 2.46 \times 10^{-5}$ m and an acceleration of ± 0.00044 g for the excitation.

The experimental model is tested under various levels of excitation spectral density. This is achieved by keeping the input signal gain constant (Master Gain on Ling Amplifier) for the range of internal detuning of the model. The level of amplification is adjusted to five levels for testing of both the first and the second normal-frequency bandwidths. Another series of tests is conducted for excitation spectral density that covers both normal-mode frequencies.

B. Experimental Results

The experimental results include sample records of time history responses and the mean square responses in terms of generalized and normal coordinates. In order to examine the stationarity of the response processes, several tests are conducted at the same excitation level, and "almost" identical graphs are obtained for the means and mean squares. Based on the fact that the response processes are almost stationary, the data processing is performed on one sample record for each internal detuning. The mean square response will be plotted against the internal detuning parameter $r = \omega_2/\omega_1$ and the excitation spectral density level. The bandwidth of the random excitation depends on the mode under investigation.

1. *First Mode Excitation*

The first mode is excited by a limited-bandwidth random excitation of bandwidth 5 Hz and a central frequency very close to the first normal-mode natural frequency. The frequency content of this random process is selected such that it does not excite any higher structural modes. For the five levels of excitation spectral density, the system response is governed mainly by the first mode, which does not show any nonlinear coupling. Figure 4 shows a sample of the time history response under excitation spectral level $S = 0.0142$ (g^2/Hz) when the model is internally tuned to the resonance condition $\omega_2/\omega_1 = 2.0$ by adjusting the position of the vertical mass

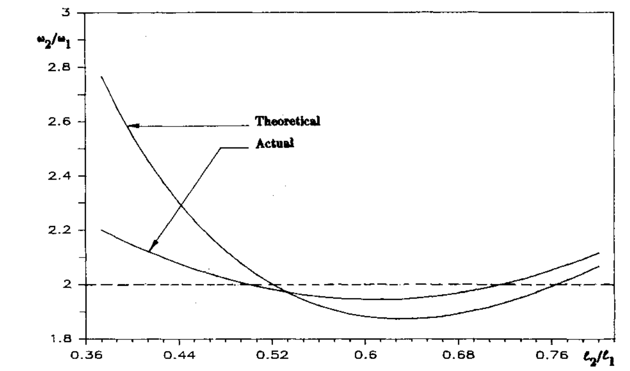


Fig. 3 Comparison of measured to theoretical frequency ratio of the model as a function of the length ratio, $\ell_1 = 0.19$ m.

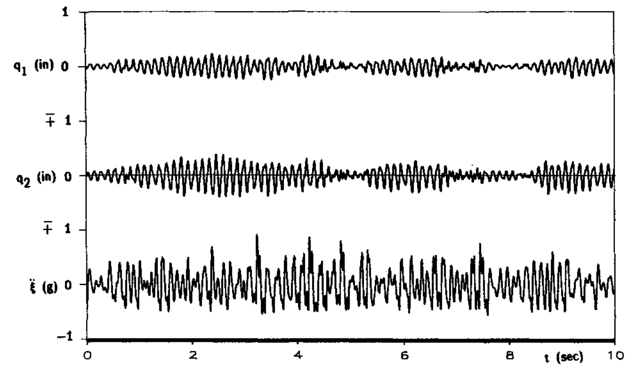
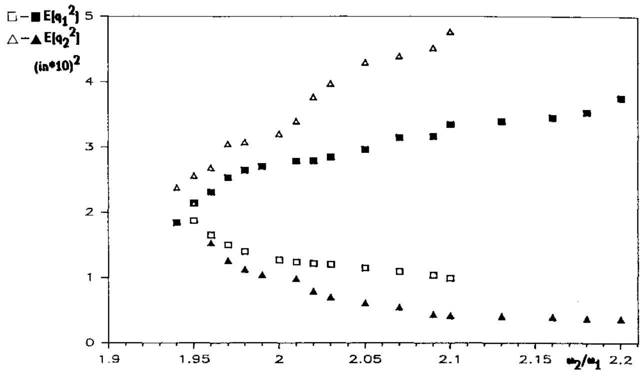
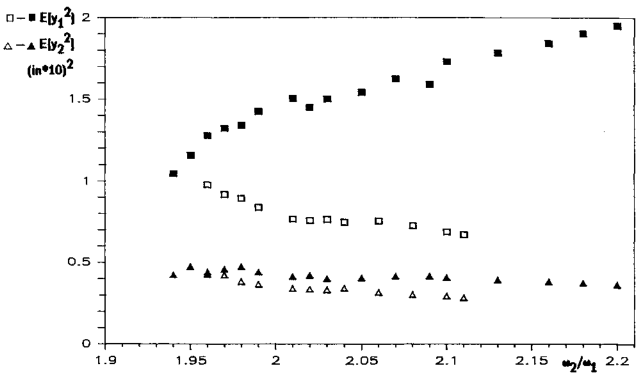


Fig. 4 Time history response first normal-mode excitation. Level V: $S_0 = 0.0142$ g^2/HZ, $\ell_2/\ell_1 = 0.71$, $m_2/m_1 = 0.74$.



a) **Generalized coordinates**



b) **Normal coordinates**

Fig. 5 Mean square response under first normal-mode excitation. Level V: $S_0 = 0.0142$ g²/HZ, □-△ $\ell_2/\ell_1 \geq 0.64$, ■-▲ $\ell_2/\ell_1 \leq 0.63$.

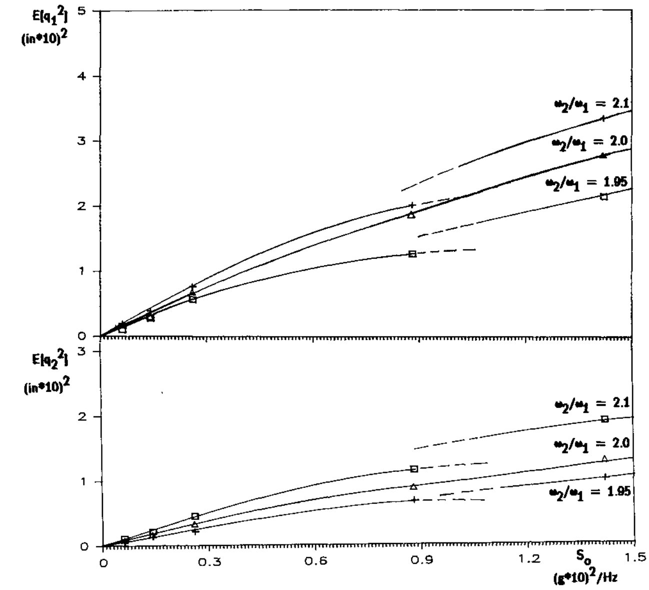
according to Fig. 3. It is seen that the response is characterized by a narrow-band random process of frequency close to the first normal mode = 7.5 Hz.

Figure 5a shows the mean square response of the generalized coordinates for the same excitation spectral density level of Fig. 4. The empty points are measured when the mass of the vertical beam moves upward while the full points are obtained when the mass moves downward. Both groups are measured in the neighborhood of the system internal resonance $r = 2$. The group of full points indicates that the mean square of the horizontal beam increases while the mean square of the vertical beam decreases as the normal-mode frequency ratio ω_2/ω_1 increases. This implies that the model behaves like a single-degree-of-freedom system for $\omega_2/\omega_2 \gg 2$. For the second group of results (empty points), the mean square response of the vertical beam increases and the mean square of the horizontal beam decreases. This feature belongs to the characteristics of linear vibration absorbers due to inertia coupling. The corresponding response curves in normal coordinates are shown in Fig. 5b. The square points (empty or full) belong to the first normal mode, which obviously predominates the response. It is also seen that as the vertical mass moves downward, the model starts to behave like a linear single-degree-of-freedom system whose mean square is given by the relationship[16]

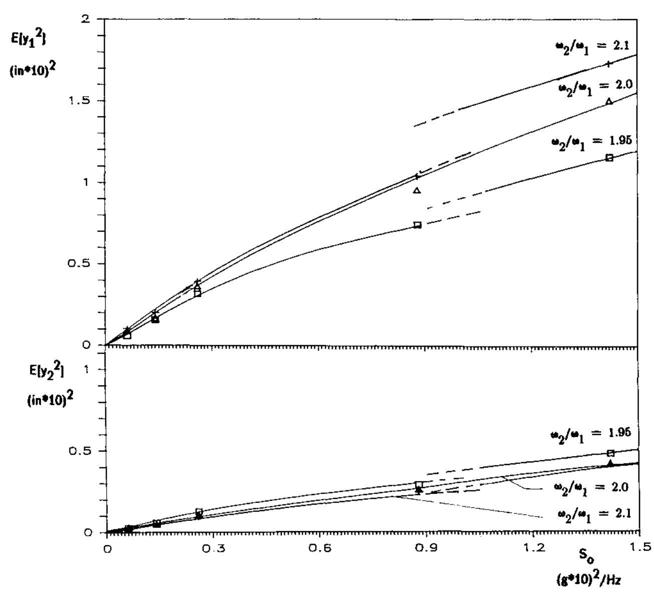
$$E[y^2] = D/\{\zeta\omega_n^3 m^2\} \tag{6}$$

where m, ω_n, and ζ are the mass, natural frequency, and damping ratio of the system, respectively. $2D$ is the excitation spectral density of a wideband random excitation. It is clear that the trend of the full square points agrees with the linear solution of Eq. (6) that the mean square response is inversely proportional to the cube of the first normal-mode frequency.

In order to provide more insight into the system response statistics, the mean square response is plotted against the excitation spectral density level as shown in Fig. 6a for various values of internal detuning. It is seen that the mean squares of the two beams increase with the excitation spectral density up to a certain level above which the curves are discontinuous. The degree of discontinuity depends on the internal detuning. Any deviation from the exact internal detuning results in a strong discontinuity. This discontinuity means that the system is unstable in the mean square sense. Similar features were reported in the deterministic response of the same system by Haddow et al.[5] The location of discontinuity is strongly dependent on the values of damping ratios and the internal detuning of the structure. Figure 6b shows the mean square response of the normal coordinates against the excitation spectral density. The curves have the same trend as Fig. 6a.



a) **Generalized coordinates**



b) **Normal coordinates**

Fig. 6 Dependence of mean square response on the excitation spectral density, $\ell_2/\ell_1 \leq 0.63$.

2. Second Mode Excitation

The second normal mode is excited by a limited-band random excitation of bandwidth 5 Hz and a central frequency

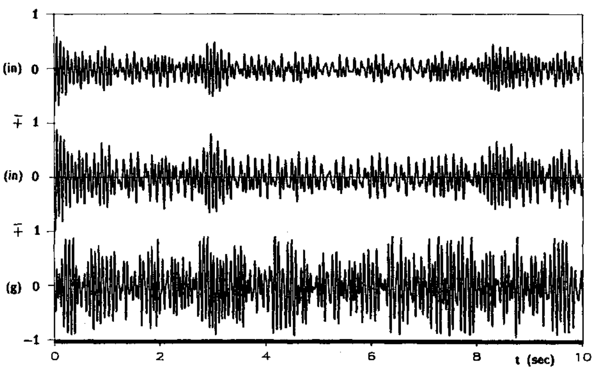


Fig. 7a Time history response of second-mode excitation.

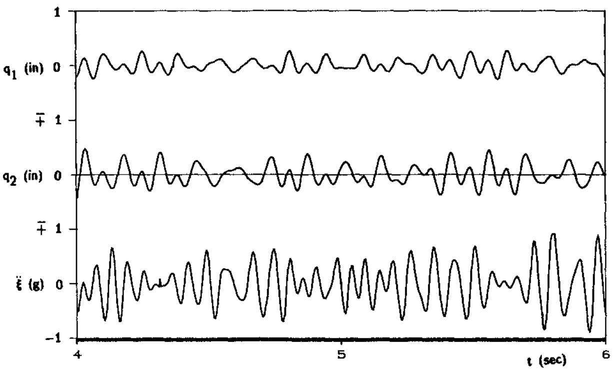
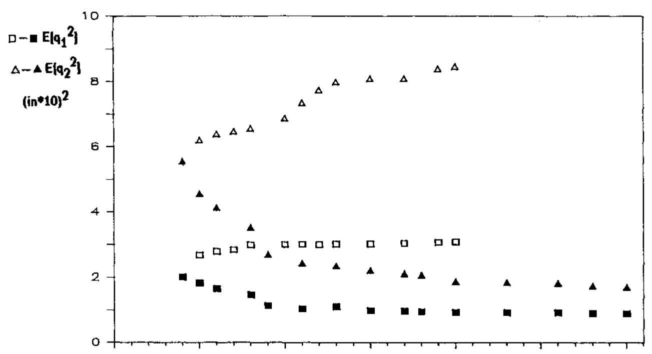
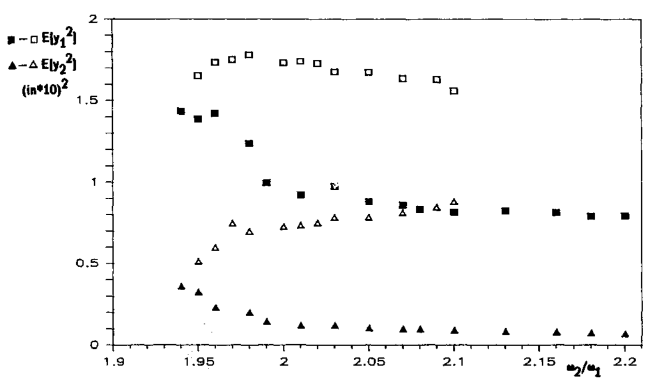


Fig. 7b Magnification of time history response of second-mode excitation. Level V: $S_0 = 0.022$ g^2/Hz, $\ell_2/\ell_1 = 0.71$, $m_2/m_1 = 0.74$.



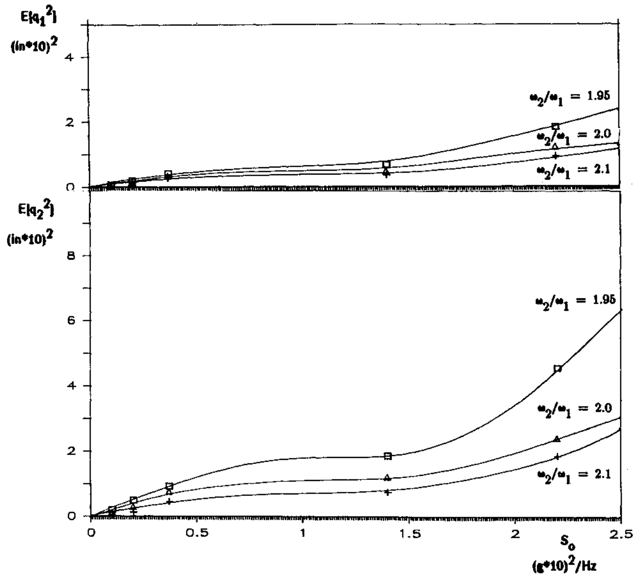
a) Generalized coordinates



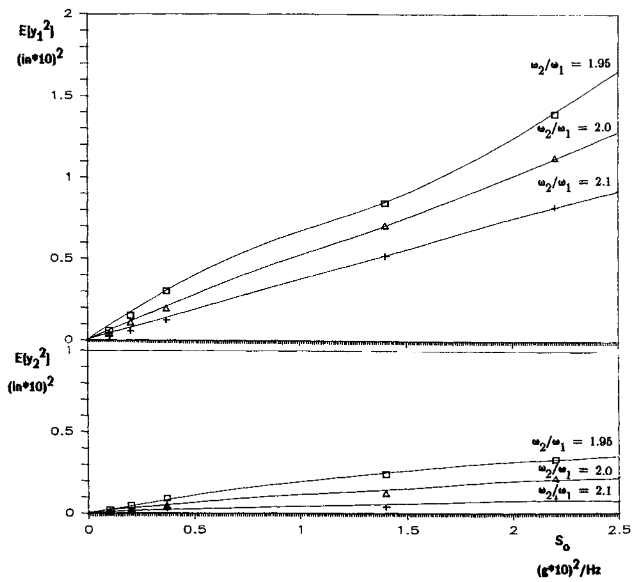
b) Normal coordinates

Fig. 8 Dependence of mean square response on internal detuning. Level V: $S_0 = 0.0142$ g^2/HZ, □-△ $\ell_2/\ell_1 \geq 0.64$, ■-▲ $\ell_2/\ell_1 \leq 0.63$.

very close to the second normal-mode frequency. Five levels of excitation spectral density ranging from 0.001 g^2/Hz to 0.022 g^2/Hz are selected. A general feature of the time history response records is that both amplitudes q_1 and q_2 increase with the levels of excitation as the first-mode excitation. The records also show that for all selected beam length ratios and all levels of excitation spectral density, the vertical beam amplitude q_1 is always greater than the horizontal beam amplitude q_2. Another observation is that when the excitation level is held constant, the amplitudes q_1 and q_2 increase slightly as the beam length ratio increases. For small levels of excitation spectral density, the second normal mode is observed to have no interaction with the first mode. However, above a certain level of excitation spectral density, it is found that the first mode appears for a certain period of time and then disappears as the second mode takes over, and so on, as shown in Fig. 7a. This nonlinear interaction of the two normal modes is clarified further in Fig. 7b. Under harmonic excitation similar energy exchange between the two modes was observed.[5] Furthermore, it was shown that the directly excited mode becomes saturated and energy is transferred to



a) Generalized coordinates



b) Normal coordinates

Fig. 9 Dependence of mean square response on excitation spectral density, $\ell_2/\ell_1 \leq 0.63$.

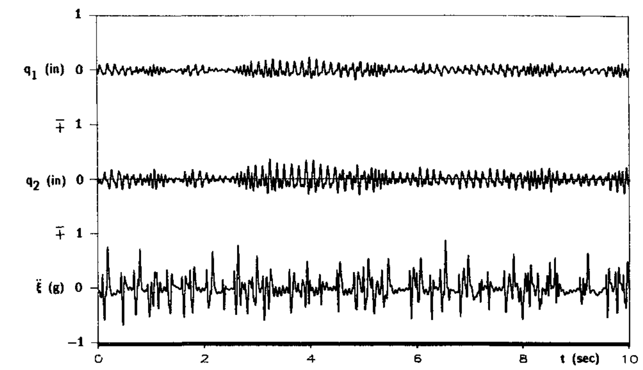
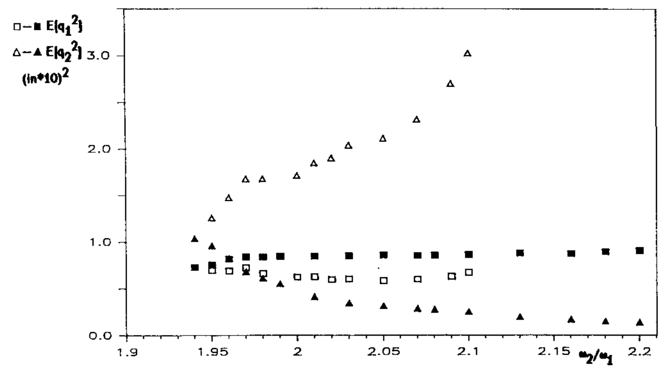
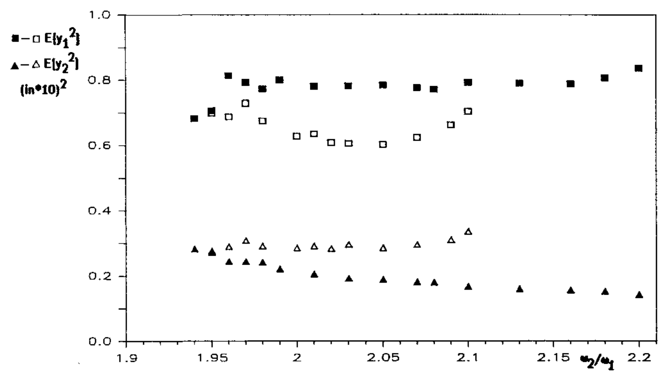


Fig. 10 Time history response of two normal-mode excitation, $\ell_2/\ell_1 = 0.71$, $m_2/m_1 = 0.74$.



a) Generalized coordinates



b) Normal coordinates

Fig. 11 Dependence of mean square response on the internal detuning under two-mode excitation, $S_0 = 0.0026$ g^2/HZ, □-△ $\ell_2/\ell_1 \geq 0.64$, ■-▲ $\ell_2/\ell_1 \leq 0.63$.

the first mode. In the present investigation, the energy transfer takes place not only under high levels of excitation spectral density but also when the internal resonance is approaching the value 2 as vertical beam length is increasing.

The mean square response of the generalized and normal coordinates are plotted against the internal detuning parameter r in Figs. 8a and 8b, respectively. The suppression effect of the excited mode takes place only when the vertical mass is moved downward, as shown in Fig. 8b by the full triangular points. The second-mode mean square (empty triangular points) increases with a corresponding decrease in the first-mode mean square (as the vertical mass moves upward).

Figures 9a and 9b show the influence of the excitation spectral density on the normal-mode mean square responses of the generalized and normal coordinates, respectively. Figure 9b indicates that the second normal-mode mean square is relatively smaller than the first normal-mode mean square response. This suppression effect is due to the nonlinear normal-mode interaction. However, the saturation phenomenon, known in deterministic systems with quadratic nonlinearity, is not pronounced in the present results since the excitation is a random process that contains several frequencies, each of which may excite the two modes. In deterministic excitation, the external and internal detunings are very important in establishing the saturation phenomenon.

3. Two-Mode Excitation

The purpose of these tests is to explore the behavior of the system under random excitation that covers both normal-mode frequencies. Due to the shaker limitation the tests are conducted under single-excitation spectral density level $S_0 = 0.0026$ g^2/Hz. A sample of the time history response record is shown in Fig. 10, which reveals the presence of the two modes. The amplitude of oscillation of each beam depends on the vertical mass location, which yields the same internal resonance condition. Figures 11a and 11b show the dependence of the mean square response on the internal detuning in terms of generalized and normal coordinates, respectively. The full points reveal linear response characteristics while the empty points show a nonlinear interaction between the two modes.

IV. Conclusions and Discussion

The results of an experimental investigation of nonlinear modal interaction in a two-degree-of-freedom structural model under random excitation are reported. The model equations of motion include linear and nonlinear inertia couplings of the generalized coordinates. The normal-mode frequencies ω_1 and ω_2 of the model are adjusted to meet the internal resonance $r = 2.0$. This frequency ratio is found to exist at two beam length ratios $\ell_2/\ell_1 = 0.49$ and 0.71. At these locations the system response characteristics are completely different when the model is excited by a band-limited random excitation. Three main tests are conducted to examine the system response behavior when the first and second modes are excited separately and when both modes are excited simultaneously.

When the first normal mode is externally excited it is found that the mean squares of the two modes increase monotonically with excitation spectral density. The response-excitation relationship is almost linear for small excitation levels. When the two beams are tuned to the exact internal resonance, the response-excitation relationship follows a continuous curve. For different internal detuning, the response curves exhibit a discontinuity. This feature is similar to the deterministic characteristics of the same model.

When the second normal mode is externally excited, the system response is dominated by the second normal mode up to an excitation spectral density level above which the first normal mode attends and the two normal modes exhibit nonlinear interaction. Above this excitation level, the first normal mode shows large random motion, that results in a suppression of the second mode. The results do not display any evidence of the existence of the saturation phenomenon. The main features of the vibration autoparametric absorber effect reported theoretically by Ibrahim and Heo[12] are not exactly confirmed in the measured results. It is believed that the deviation from theory is attributed to the fact that the experimental excitation is a band-limited random process, while in theory it is represented by a wideband random process. Another source of deviation is that the transformation into normal coordinates is not exact since it does not take into account the effect of structural damping. To eliminate this problem the first author is undertaking another series of experimental tests by using other models whose generalized and normal coordinates are the same. The results of the new experiments will be processed by using advanced data acquisition equipment.

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